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Charnley AK (1992). Mechanisms of fungal pathogenesis in insects with particular reference to locusts. In: Lomer CJ, Prior C (eds) Biological Controls of Locusts and Grasshoppers: Proceedings of an international workshop held at Cotonou, Benin. Oxford: CAB International, pp 181-190.

Mundree SG, Farrant JM (2000). Some physiological and molecular insights into the mechanisms of desiccation tolerance in the resurrection plant Xerophyta viscasa Baker. In Cherry et al. (eds) Plant tolerance to abiotic stresses in Agriculture: Role of Genetic Engineering, Kluwer Academic Publishers, Netherlands, pp 201-222.

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Full Length Research Paper

Simulation of 4π HPGe Compton-Suppression spectrometer

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Compton-suppression spectrometer is well suited to the analysis of low levels of radioactive nuclides. Monte Carlo simulations can be a powerful tool in calibrating these types of detector systems, provided enough physical information on the system is known. A simplified Compton-suppression spectrometer model using the Geant–4 simulation toolkit was discussed. The spectrometer model was tested to evaluate photo peak efficiency of detecting point and disk sources. The efficiency calibration was calculated for incident gamma energy from 200 to 3000 keV in both the suppressed and unsuppressed mode of operation. The applicability of the efficiency transfer method in various measurement geometries was tested successfully. It can save time and avoid tedious experimental calibration for different samples geometries.

Key words: Compton-suppression spectrometer, photopeak efficiency, Geant–4 Monte Carlo simulation.

INTRODUCTION

Gamma ray spectrometry based on high pure germanium (HPGe) detectors is an important tool in the field of radioactivity measurements. The reason is due to the excellent energy resolution of HPGe detectors that permits the analyses of various radionuclides in composite samples selectively as well as the high efficiency of recently produced HPGe detectors (L'Annunziata, 2012).

Improved detection system for experiments need low background environment and is strongly needed especially in the field of high energy physics; dark matter, low-energy solar neutrino experiments. Compton-suppression spectrometer is considered a powerful technique to reduce the contribution from Compton scattered photons in a measured sample. Generally, it consists of primary detector surrounded by secondary detectors, and the pulse produced by the primary detector is accepted by analyzer only when the secondary detectors do not produce a pulse within a time period. The most wide Compton-suppression spectrometer consists of high pure germanium (HPGe) detector surrounded by scintillator crystals such as CsI (Tl), NaI(Tl) or BGO in 4π geometry coupled by photomultipliers (PMTs) as in Exogam, Miniball, Gamma-Sphere, Euroball, GASP. By operating spectrometer in a fast anticoincidence regime (tens of nanoseconds time resolution) can be obtained as a significant reduction of the Compton background with high resolved spectra (Fan et al., 2013; Breier and Povinec, 2009 and detectors websites).

Samples suitable to be measured are generally disk sources (various geometries and matrices) which would require certain measurements. The spectrometer accuracy is essential for the absolute measurements of radioactive materials. Experimental measurements can be applied to limited different geometries, compositions

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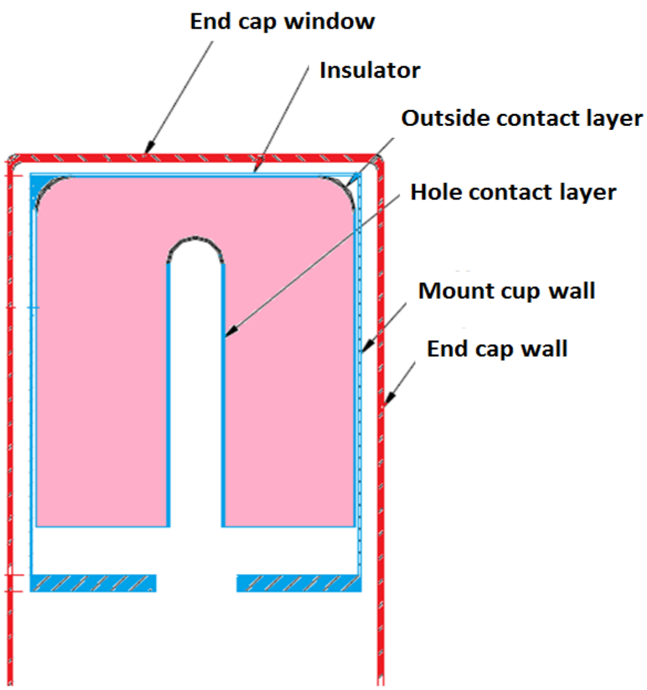

Figure 1. Schematic illustration of the HPGe detector.

and densities of sources and cannot be applied directly to all configurations. This can be a time consuming process and in some cases impossible to replicate. As an alternative approach, Monte Carlo simulations are recommended to do this task and then to continue with experimental measurements. Monte Carlo simulation is a powerful and flexible tool for simulating various physical phenomena. These types of simulations can be powerful complement to experimental installations. Modeling the geometry in computer environment gives flexibility and ease of use instead of performing experiment in different geometries (Britton, 2012; McNamara et al., 2012; Rehman et al., 2011).

One of the main problems faced in any detection system is evaluation of detection efficiency. So, instead of performing efficiency calibration for each sample, which of course is impractical, the model can be used to calculate the efficiency of the spectrometer for that particular geometry. This work aims to present a preliminary evaluation of detection efficiency of Compton-suppression spectrometer in both of the suppressed and unsuppressed mode using Geant–4Monte Carlo simulation based on object-oriented methodology and C++ language (Agostinelli et al., 2003).

SPECTROMETER SIMULATION

Geant–4Toolkit

Geant–4 is a simulation tool kit that can simulate accurately the detectors and interactions of photon and particles through matters. The code was written in C++ and developed by CERN (CERN, 2007). It simulates accurately the passage of particles through matter. It offers the possibility to include a complete description of an experiment and extract information that might be useful in many different fields, such as nuclear and high energy physics, medical physics and astrophysics.

Geant–4 toolkit contains a complete range of functionalities including tracking, geometry; physics models and hits. It is controlled through the instantiation of the appropriate Geant4 classes to define the geometry, applicable physics, and particle source and to control the execution. The key class for all Geant4 applications is the G4RunManager which controls the initializations of geometry, physics list and primary particle generation. The user has full freedom to develop an own simulation program. The user must implement several mandatory classes to describe the detector geometry, the primary particle generator and a class to describe the relevant particles and physics processes. Other non-mandatory classes must be created to resolve proper objectives.

Simulation of spectrometer geometry

Modeling the spectrometer geometry in computer environment gives flexibility and ease of use, instead of performing an experimental determination of detection efficiency for different geometries. For this reason, the model of spectrometer system would be useful for further experiments when the geometry is changed. Then, instead of performing efficiency calibration for each sample, which may sometimes be impractical, the model can be used to calculate the efficiency of the system for that particular geometry (Chirosca et al., 2013; Baccouche et al., 2012).

The supposed Compton-suppression spectrometer geometry HPGe detectors surrounded by a CsI(Tl) cylindrical scintillator crystal with photomultipliers attached. The simulated detector is p coaxial type detector. A sketch of the HPGe detector, adapted from the manufacturer manual, is shown in Figure 1 and the provided dimensions of the HPGe detector are further summarized in Table 1.
The HPGe cylinder is oriented inside a scintillator cylinder with diameter 150 mm and length 250 mm. Sample with diameter 5 mm and length 15 mm is placed at distance 10 mm from the top of the HPGe crystal. The detection system of germanium crystal works in anticoincidence with photomultipliers, trying to eliminate a part of Compton electrons generated by gamma rays which are not absorbed by photoelectric effect in HPGe crystal. The proposed Geant–4code was written using ten user classes: three user mandatory classes and some user action classes. The geometry of the Compton-suppression spectrometer was coded in the mandatory class (Detector Construction). For this study, physics process was defined in the other mandatory class (Physics List). For gamma rays, Compton scattering, photo electric absorption, pair production and Rayleigh (coherent) scattering processes are defined with valid energy range down to 250 eV. The definition is also disk to include electrons and positron multiple scattering, ionization and bremmsstrahlung processes. Atomic effects after photoelectric effect, as X-rays emission and Auger effect are included. Two Sensitive Detector classes, one for HPGe detector and one for CsI(Tl) scintillator were built. Registration of interesting processes is made in Stepping Action class. Algorithm for generating computing efficiency was implemented in (Run Action) class.

RESULTS AND DISCUSSION

The full model of efficiency calculation and efficiency transfer has been written in ten user classes: three user mandatory classes and some user action classes are

Table 1. Summary of the components and dimensions of the HPGe detector provided by the manufacturer.

Component	Dimension (mm)
Ge crystal diameter	69.8
Ge crystal length	89.5
Core hole diameter	11.6
Core hole depth	80.8
Outer electrode thickness	0.5
Inner electrode thickness	0.003
Window electrode thickness	0.003
Endcap window thickness	0.76
Aluminium endcap thickness	1.5

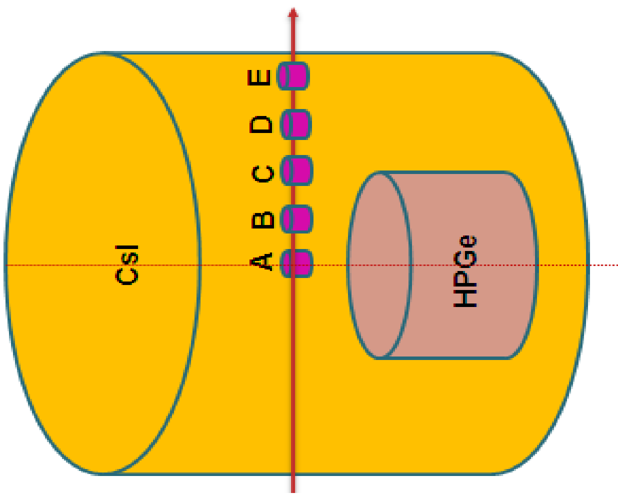

Figure 2. Detected source at different positions.

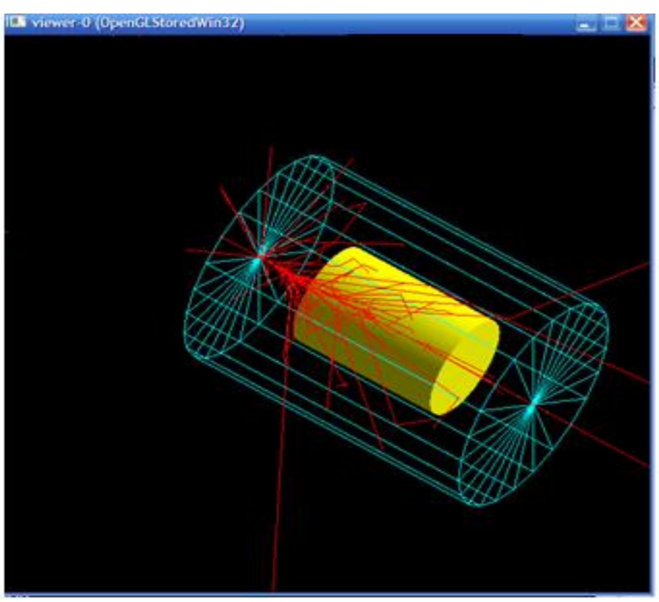

Figure 3. Visualization model of the detection system.

written. Registration of interesting processes is made in Stepping Action class. In Run Action class the simplified algorithm for computing efficiency was implemented. The detector system was modelled in both the suppressed and unsuppressed modes. In the unsuppressed mode, the coincidence registering (sensitive detector) of the surrounding CsI is turned off. While for the suppressed mode, a photon hit in the HPGe detector is triggered if a hit in the NaI detector does not occur within the same event. Both the HPGe and CsI detectors were set as sensitive detectors in the model and photon hits in each were collected using the G4HCofThisEvent class.

The principle of calculating full-energy peak efficiency is started by initialization of the decay process; then the deposited energy of each photon in the detector is summed after completing the photon tracking. The tracking of a single photon is stopped when it leaves the volume of interest or when the energy of photon becomes lower than a specified threshold value called cut-off energy. Consequently, a realistic spectrum of energy deposited in the detectors is obtained through simulation and the experimental efficiency can be compared to the simulated one so-called apparent efficiency obtained by calculating the peak area, after correcting for the continuum to match the measured one.

The efficiency of the investigated Compton suppression γ-spectrometer was generated using Geant-4 model for both isotropical point source and volume source, placed at five different positions (A, B, C, D, E) of the spectrometer as shown in Figure 2. The visualization model of the investigated Compton suppression γ-spectrometer generated by Geant-4 is shown in Figure 3. Total of 10^6 events for each energy for each different source positions, from 200 to 3000 keV, were utilized to determine system efficiency. The summation effects are also considered a source of errors in the simulated efficiency values. This effect appears at ^{60}Co (1173.2, 13323.5 keV). Geant 4 code has been applied to correct these summation and to improve their values of peak efficiencies.

The first round of simulations was performed using a

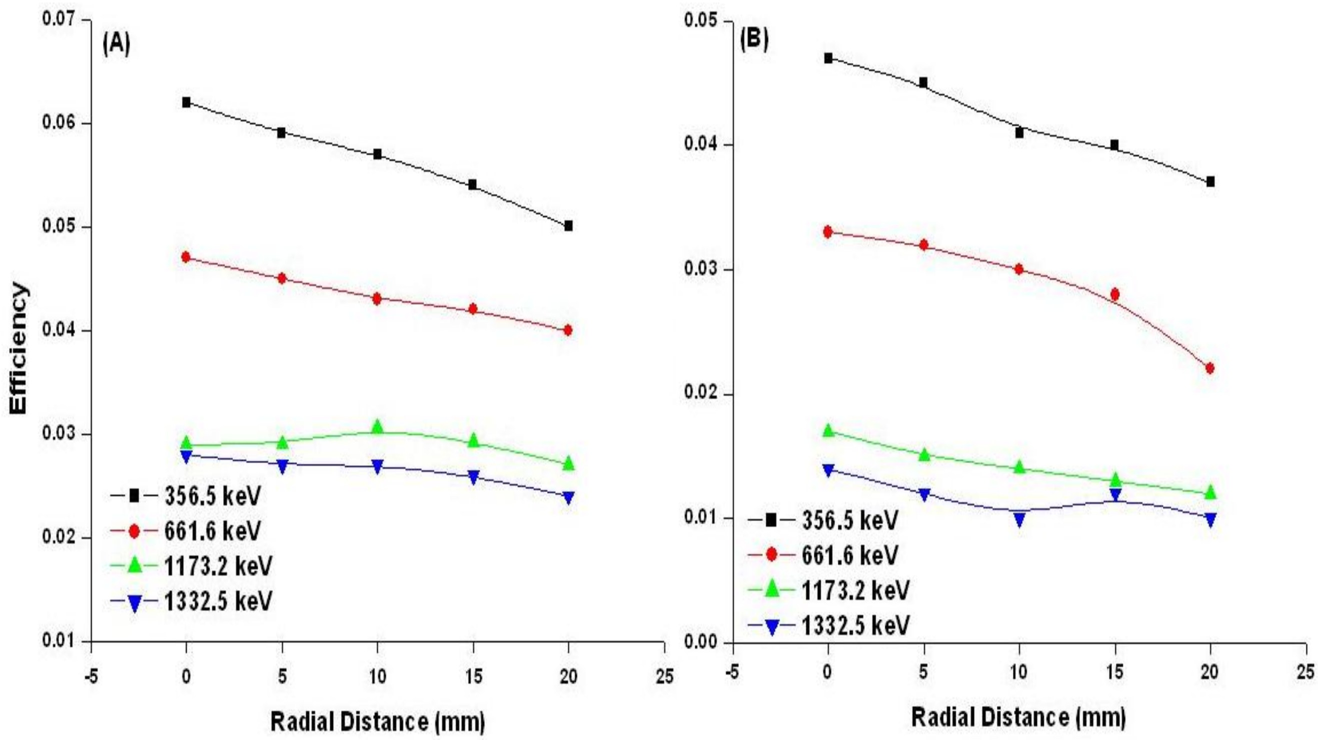

Figure 4. Variation of simulated photopeak efficiency for a point source at various positions for Compton suppression γ-spectrometer:- A) without CsI(Tl) anticoincidence scintillator (anti-suppressed mode); B) with CsI(Tl) anticoincidence scintillator (suppressed mode).

point source. Variation of photo-peak efficiency in cases of including CsI (Tl) scintillator as a suppressed mode to the detection system at different positions is shown in Figure 4. The trend is same for various gamma ray energies (356.5; 661.6; 1173.2; 1332.5) but is more pronounced for lower gamma ray energies than higher energies as presented in Table 2.

The second round of simulations was performed using a disk source. The variation of simulated efficiency in the two spectrometer cases is shown in Figure 5 and presented in Table 3. It is clear from Tables 2 and 3 that the efficiency in the suppressed mode is lower. This is largely due to the coincidence detection of two different photopeak photons from the same source, in a time frame shorter than the timing resolution of the detector system.

The applicability of the efficiency transfer in various measurement geometries on the basis of the simulated efficiency for reference point source geometry can be applied successfully. The detector efficiency was calculated for the same locations of the point sources and also for the disk sources. Figure 6 shows relation between point and disk source. It is clear that there is an increase at 1000 to 14500 keV in comparison with the fitting. It returns to the coincidence summing effects of ^{60}Co (1173.2, 13323.5 keV) for the transformation between point and bulk samples. The transfer efficiency is computed for discreet values of the fitted efficiency data of point source to derive new efficiencies values for a disk source which can be applied successfully to transfer efficiency data between the two geometries as presented in Table 4.

Conclusion

Monte Carlo simulation is a powerful tool in designing and calibrating different types of detection system and spectrometers. The applicability of this method is greatly dependent on the accuracy of detector geometry model. Additionally, the geometry, composition and density distribution of the sample matrix may be particularly important in models used for low level background. The Geant-4simulation toolkit was used to model a simplified Compton-suppression spectrometer operating in both the suppressed and unsuppressed mode to determine the photo peak efficiencies incident gamma energy in a range 200 to 3000 keV. The simulations show that the efficiency in the suppressed detector mode is lower than anti-suppression mode. This is largely due to the coincidence detection of two different photons from the same source which is not within the analyzer time period. The simulated efficiency values for point as well as for disk sources at different position can be mathematically

Table 2. Simulated photopeak efficiency for a point source at various positions.

Energy (keV)	Without CsI(Tl)	With CsI(Tl)
Source position: 0 mm (A)		
356.5	0.062	0.047
661.6	0.047	0.033
1173.2	0.029	0.017
1332.5	0.028	0.014
Source position: 5 mm (B)		
356.5	0.059	0.045
661.6	0.045	0.032
1173.2	0.029	0.015
1332.5	0.027	0.012
Source position: 10 mm (C)		
356.5	0.057	0.041
661.6	0.043	0.030
1173.2	0.032	0.014
1332.5	0.027	0.010
Source position: 15 mm (D)		
356.5	0.054	0.040
661.6	0.042	0.028
1173.2	0.031	0.013
1332.5	0.026	0.012
Source position: 20 mm (E)		
356.5	0.050	0.037
661.6	0.040	0.022
1173.2	0.030	0.012
1332.5	0.024	0.010

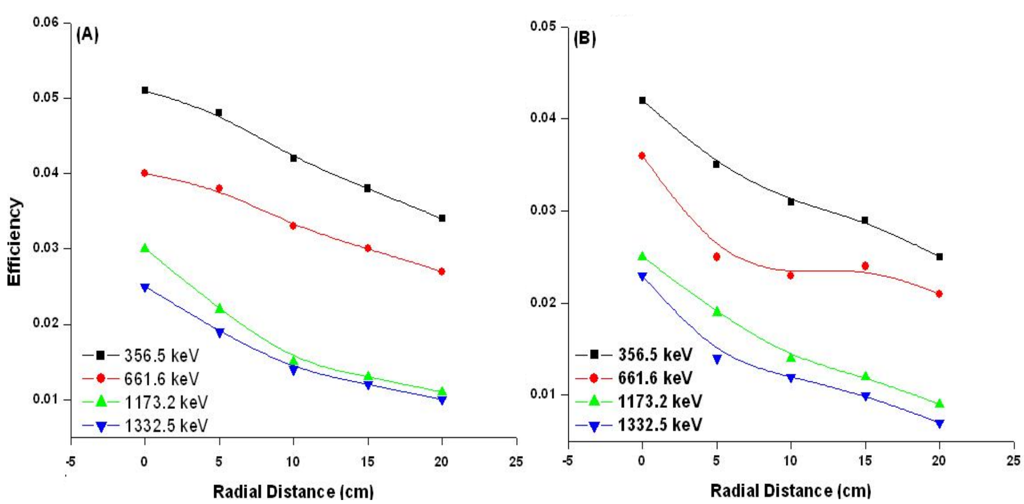

Figure 5. Variation of simulated photopeak efficiency for a disk source at various positions for Compton suppression γ-spectrometer:- A) without CsI(Tl) anticoincidence scintillator (anti-suppressed mode)., B) with CsI(Tl) anticoincidence scintillator (suppressed mode).

Table 3. Simulated photopeak efficiency for disk source at various positions.

Energy (keV)	Without CsI(Tl)	With CsI(Tl)
Source position: 0 mm (A)		
356.5	0.051	0.042
661.6	0.040	0.036
1173.2	0.030	0.025
1332.5	0.025	0.023
Source position: 5 mm (B)		
356.5	0.048	0.035
661.6	0.038	0.025
1173.2	0.022	0.019
1332.5	0.019	0.014
Source position: 10 mm (C)		
356.5	0.042	0.031
661.6	0.033	0.023
1173.2	0.015	0.014
1332.5	0.014	0.012
Source position: 15 mm (D)		
356.5	0.038	0.029
661.6	0.030	0.024
1173.2	0.013	0.012
1332.5	0.012	0.010
Source position: 20 mm(E)		
356.5	0.034	0.025
661.6	0.027	0.021
1173.2	0.011	0.009
1332.5	0.010	0.007

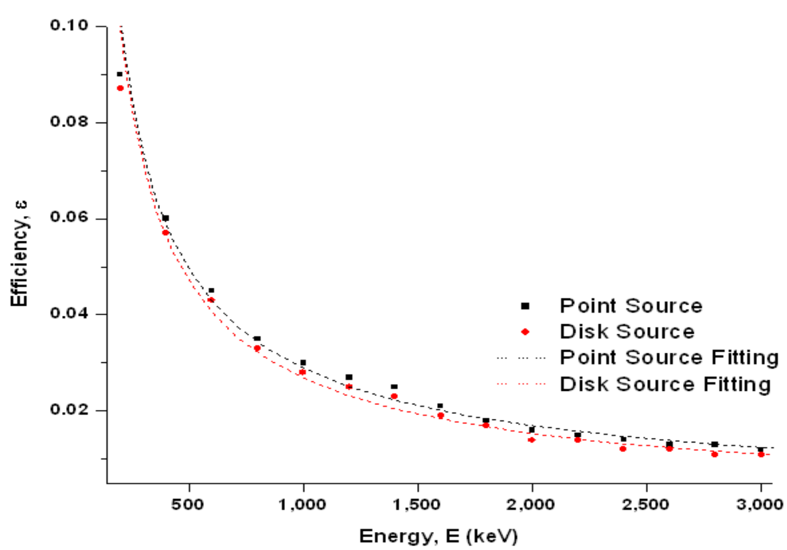

Figure 6. Mathematical efficiency transfer relation between point and disk sources.

Table 4. Photopeak efficiency transfer between point and disk sources.

Energy (keV)	Point →Disk		Disk → Point	
	Obs.	Calc.	Obs.	Calc.
356.5	0.061	0.058	0.062	0.064
661.6	0.045	0.043	0.047	0.048
1173.2	0.027	0.026	0.029	0.029
1332.5	0.025	0.025	0.028	0.027

related for the same locations to save time and avoid experimental calibration for different samples geometries.

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Full Length Research Paper

Vectorial reduced differential transform (VRDT) method for the solution of inhomogeneous Cauchy-Riemann system

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It is well known that initial-value problem for the Cauchy-Riemann system is ill-posed and the problem with such Hadamard instability cannot be solved unless the initial data are analytic. In this paper, we present the vectorial reduced differential transform (VRDT) method to solve initial-value problem for the inhomogeneous Cauchy-Riemann system with analytic data. The VRDTM solution vector achieved is in the form of infinite series whose compact form is in agreement with the exact solution vector.

Key words: Inhomogeneous Cauchy-Riemann system, initial-value problem, vectorial reduced differential transform.

INTRODUCTION

This paper deals with the system of first-order linear equations:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}, \qquad x \in \mathbb{R}, y > 0, \tag{1}$$

for the desired vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ involving real-valued functions $u(x,y)$ and $v(x,y)$. Collectively, System (1) is elliptic while individually both the partial differential equations are hyperbolic for the ellipticity of the system we refer to Wendland (1979). If $f(x,y) \equiv 0 \equiv g(x,y)$, Equation (1) is the Cauchy-Riemann system and dependent variables u, v are analytic. Thinking of y as a time variable and of data for the vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ as being given on $y = 0$, we are mainly concerned with the inhomogeneous Cauchy-Riemann System (1) subject to the following initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \quad x \in \mathbb{R}, \tag{2}$$

where $\phi(x)$ and $\psi(x)$ are analytic.

It is well known that initial value problem for the Cauchy-Riemann system is ill-posed. The inherent instability of this system, for the first time was discussed by Hadamard (1923). Farmer and Howison (2006) illustrate the ill-posed nature of the system in various contexts.

The paper of Joseph and Saut (1990), which is the main source of motivation to our present work, associates the ill-posedness of Cauchy problem with the non-existence of solution to the initial-value problem for non-analytic data. They show that the problems which are Hadamard unstable cannot be solved unless the initial data are analytic. Reichel (1986) analyses several fast numerical methods based on solving initial-value problems for the Cauchy-Riemann system. She discusses the techniques for analytic continuation of

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conformal mappings and indicates the available methods for finding analytic continuations which use Taylor coefficients or their approximations for the analytic functions, see for example Baker and Graves-Morris (1981), Gustafson (1978), Niethammer (1977) and Henrici (1966). Reichel (1986) also shows the stability and accuracy of her schemes through numerous applications.

During the past couple of decades, researchers have been engrossed to constructing the approximate analytic solution for the partial differential equations. Zhou (1986) introduced the differential transform method. To solve Cauchy problem for various PDEs, the reduced differential transform method has recently been used by Keskin and Oturance (2009), Keskin (2010), Cenesiz et al. (2010), Taha (2011) and Hesam et al. (2012).

In this paper, we present vectorial reduced differential transform (VRDT) method to solve the initial-value problem for the inhomogeneous Cauchy-Riemann system. The method is applied in various situations of Cauchy-Riemann system with a variety of initial data. The VRDTM solutions are in the form of infinite series whose compact forms are in agreement with exact solutions.

VECTORIAL REDUCED DIFFERENTIAL TRANSFORM (VRDT)

Definition 1

Let $u(x,y)$ be an analytic function (obviously sufficiently smooth with respect to *x* and *y* in the domain of definition). The reduced differential transform $U_k(x)$ of $u(x,y)$ is defined as (Keskin and Oturance, 2009):

$$U_k(x) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}\, u(x,y)\right\}_{y=0}. \quad (3)$$

The inverse-RDT, $u(x,y)$ of $U_k(x)$ is defined as:

$$u(x,y) = \sum_{k=0}^{\infty} y^k U_k(x). \quad (4)$$

From (3) and (4) the following result is obtained:

$$u(x,y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}\, u(x,y)\right\}_{y=0}. \quad (5)$$

The basic RDTs are given in Table 1 and can be proved using Definitions (3) and (4), see for the details (Keskin, 2010).

The definition of reduced differential transform can be extended to the vectors of analytic functions which are given as follow:

Definition 2

Let $[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T$ denotes a column vector with elements as analytic functions (obviously sufficiently smooth with respect to x, y in the domain of definition). The vectorial reduced differential transform (VRDT) of the vector $[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T$, given by the vector $[U_k(x) \quad V_k(x) \quad W_k(x) \quad \cdots]^T$, is defined as:

$$[U_k(x) \quad V_k(x) \quad W_k(x) \quad \cdots]^T = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T\right\}_{y=0} \quad (6)$$

The inverse-VRDT vector, $[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T$ of vector $[U_k(x) \quad V_k(x) \quad W_k(x) \quad \cdots]^T$ is further defined by:

$$[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T = [\textstyle\sum_{k=0}^{\infty} y^k U_k(x) \quad \sum_{k=0}^{\infty} y^k V_k(x) \quad \sum_{k=0}^{\infty} y^k W_k(x) \quad \cdots]^T. \quad (7)$$

The following result is immediately obtained from (6) and (7).

$$[u(x,y) \quad v(x,y) \quad w(x,y) \quad \cdots]^T = \left[\sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}u(x,y)\right\}_{y=0} \quad \sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}v(x,y)\right\}_{y=0} \quad \sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}w(x,y)\right\}_{y=0} \quad \cdots\right]^T \quad (8)$$

VRDT METHOD TO SOLVE INITIAL VALUE PROBLEM FOR THE INHOMOGENEOUS CAUCHY-RIEMANN SYSTEM

A proper posing of initial-value problem for the Cauchy-Riemann system is very important since the existence of a unique solution which is also guaranteed by the Cauchy-Kowalevsky theorem (Walter, 1985), may not continuously depend upon initial data (Hadamard, 1923). Joseph and Saut (1990) show that the problems with Hadamard instability cannot be solved unless the initial data are analytic.

To solve the initial-value Problem (2) with analytic data functions, for the inhomogeneous Cauchy-Riemann System (1), the VRDT method proceeds as follows. The VRDT on Problem (1) and (2) yields the following recurrence relations:

$$(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k + \begin{bmatrix}F(x)\\G(x)\end{bmatrix}_k, \quad (9)$$

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0 = \begin{bmatrix}\phi(x)\\\psi(x)\end{bmatrix}, \quad (10)$$

where the vectors $(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1}$, $\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k$ and $\begin{bmatrix}F(x)\\G(x)\end{bmatrix}_k$ represent the VRDTs for the vectors $\frac{\partial}{\partial y}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}$, $\frac{\partial}{\partial x}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}$ and $\begin{bmatrix}f(x,y)\\g(x,y)\end{bmatrix}$, respectively.

Table 1. Basic reduced differential transforms.

S/N	Function	Reduced Differential Transform
1.	$u(x,y)$	$U_k(x) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}\, u(x,y)\right\}_{y=0}$
2.	$u(x,y) \pm v(x,y)$	$U_k(x) \pm V_k(x)$
3.	$\alpha\, u(x,y)$	$\alpha\, U_k\,(x)$, α being a constant
4.	$u(x,y)\; v(x,y)$	$\sum_{r=0}^{k} U_r(x)\, V_{k-r}(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x)$
5.	$x^m y^n\, u(x,y)$	$x^m U_{k-n}(x)$
6.	$x^m y^n$	$x^m\, \delta(k-n), \quad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
7.	$\frac{\partial^n}{\partial x^n}\, u(x,y)$	$\frac{\partial^n}{\partial x^n} U_k(x)$
8.	$\frac{\partial^r}{\partial y^r}\, u(x,y)$	$(k+1)(k+2)\ldots(k+r)U_{k+r}(x)$

Equations (9) and (10) straightaway produce all vectors $\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_k$ iteratively. The inverse-VRDTs of the set of vectors $\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0$, $\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1$, $\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2$, …, $\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_n$ then give the VRDTM solution vector $\begin{bmatrix} \tilde{u}(x,y) \\ \tilde{v}(x,y) \end{bmatrix}$ in the form:

$$\begin{bmatrix} \tilde{u}(x,y) \\ \tilde{v}(x,y) \end{bmatrix}_n = \begin{bmatrix} \sum_{k=0}^{n} y^k U_k(x) \\ \sum_{k=0}^{n} y^k V_k(x) \end{bmatrix}, \qquad (11)$$

where n is the order of approximation for the solution. The exact solution vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ of Problems (1) and (2) then becomes:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \lim_{n\to\infty} \tilde{u}_n(x,y) \\ \lim_{n\to\infty} \tilde{v}_n(x,y) \end{bmatrix}. \qquad (12)$$

IMPLEMENTATION OF VRDT METHOD

The method is implemented on a variety of initial-value problems for the homogeneous and inhomogeneous Cauchy-Riemann systems. The data of the model problems is given in the Table 2.

Model Problem-A

In this type of model problem, we consider the homogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \qquad (13)$$

subject to the initial condition

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh x \end{bmatrix}, x \in \mathbb{R}, \qquad (14)$$

having exact solution vector given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sin y \cosh x \\ \cos y \sinh x \end{bmatrix}. \qquad (15)$$

The VRDT on (13) and (14) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k, \qquad (16)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \sinh x \end{bmatrix}, \qquad (17)$$

and iterative relations (16) and (17) yield the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \sinh x \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} \cosh x \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} 0 \\ \frac{-1}{2!}\sinh x \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} \frac{-1}{3!}\cosh x \\ 0 \end{bmatrix}, \ldots \qquad (18)$$

Table 2. Data of model problems at a glance.

Prob.	$f(x,y)$	$g(x,y)$	$\phi(x)$	$\psi(x)$
A	0	0	0	$\sinh x$
B	0	0	$\sin(x)$	$\cos(x)$
C	$(a^2-1)e^{-y}\sin(ax)$	0	$\sin(ax)$	$a\cos(ax)$
D	$(a+b)\sin(ax)\cos(by)$	$(a-b)\cos(ax)\sin(by)$	0	$\cos(ax)$
E	$4y^3\sin(4x)-2(1-x)\sin(4y)$	$4x(2-x)\cos(4y)+4y^4\cos(4x)$	0	0
F	$4y^3\sin(x)$	$4y^3+y^4\cos(x)$	0.001	0.001

Using inverse-VRDTs of (18), we obtain the VRDTM solution vector:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\,y^k \\ \sum_{k=0}^{\infty} V_k(x)\,y^k \end{bmatrix} = \begin{bmatrix} \left\{y-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots\right\}\cosh x \\ \left\{1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots\right\}\sinh x \end{bmatrix}, \quad (19)$$

whose compact form takes the form:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \sin y \\ \sinh x \cos y \end{bmatrix},$$

which is the same as exact solution vector.

Model Problem-B

We consider the homogeneous Cauchy-Riemann system (13) subject to initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (20)$$

having the exact solution vector given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(x) \\ e^{-y}\cos(x) \end{bmatrix}. \quad (21)$$

Now we use the VRDT method to solve this problem. The VRDT on (13), (20) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k, \quad (22)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}. \quad (23)$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} -\sin(x) \\ -\cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2!}\sin(x) \\ \frac{1}{2!}\cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{1}{3!}\sin(x) \\ -\frac{1}{3!}\cos(x) \end{bmatrix}, \ldots .$$

Finally the inverse-VRDTs of the above vectors give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\,y^k \\ \sum_{k=0}^{\infty} V_k(x)\,y^k \end{bmatrix} = \begin{bmatrix} \left\{1-y+\frac{y^2}{2!}-\frac{y^3}{3!}+\ldots\right\}\sin(x) \\ \left\{1-y+\frac{y^2}{2!}-\frac{y^3}{3!}+\ldots\right\}\cos(x) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(x) \\ e^{-y}\cos(x) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-C

We consider the following inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} (a^2-1)e^{-y}\sin(ax) \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \quad (24)$$

where *a* is a given parameter, subject to initial condition

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (25)$$

having the exact solution vector given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(ax) \\ a\,e^{-y}\cos(ax) \end{bmatrix}. \quad (26)$$

It is interesting to note that the problem takes the form of model problem-B if $a = 1$. The VRDT on (24) and (25) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} (a^2-1)\sin(ax)\left\{\delta(k)-\delta(k-1)+\frac{1}{2!}\delta(k-2)-\ldots\right\} \\ 0 \end{bmatrix}, \quad (27)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}. \tag{28}$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} -\sin(ax) \\ -a\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2!}\sin(ax) \\ \frac{a}{2!}\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{1}{3!}\sin(ax) \\ -\frac{a}{3!}\cos(ax) \end{bmatrix}, \dots.$$

Finally the inverse-VRDTs of the above vectors give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\}\sin(ax) \\ a\{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\}\cos(ax) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(ax) \\ a\, e^{-y}\cos(ax) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-D

For the inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} (a+b)\sin(ax)\cos(by) \\ (a-b)\cos(ax)\sin(by) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \tag{29}$$

where a, b are given parameters, we prescribe the following initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{30}$$

whose exact solution vector is given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sin(ax)\sin(by) \\ \cos(ax)\cos(by) \end{bmatrix}. \tag{31}$$

Now we use VRDT method to solve this initial-value problem. The VRDT on (29) and (30) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} (a+b)\sin(ax)\left\{\delta(k) - \frac{b^2}{2!}\delta(k-2) + \dots\right\} \\ (a-b)\cos(ax)\left\{b\delta(k-1) - \frac{b^3}{3!}\delta(k-3) + \dots\right\} \end{bmatrix}, \tag{32}$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}. \tag{33}$$

The above iterative relations then yield the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} b\sin(ax) \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} 0 \\ -\frac{b^2}{2!}\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{b^3}{3!}\sin(ax) \\ 0 \end{bmatrix}, \dots \tag{34}$$

Finally the inverse-VRDTs of vectors in (34) give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \{by - \frac{(by)^3}{3!} + \frac{(by)^5}{5!} - \dots\}\sin(ax) \\ \{1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \dots\}\cos(ax) \end{bmatrix} = \begin{bmatrix} \sin(ax)\sin(by) \\ \cos(ax)\cos(by) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-E

In this problem, VRDT method is applied on inhomogeneous Cauchy-Riemann system for the homogeneous initial data. We consider the following inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} 4y^3\sin(4x) - 2(1-x)\sin(4y) \\ 4x(2-x)\cos(4y) + 4y^4\cos(4x) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \tag{35}$$

subject to initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}, \tag{36}$$

having exact solution vector given as:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\sin(4y) \end{bmatrix}. \tag{37}$$

Now we use the VRDT method to solve this problem. The VRDT on (35) and (36) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} 4\sin(4x)\,\delta(k-3) - 2(1-x)\left\{4\delta(k-1) - \frac{4^3}{3!}\delta(k-3) + \cdots\right\} \\ 4x(2-x)\left\{\delta(k) - \frac{4^2}{2!}\delta(k-2) + \cdots\right\} + 4\cos(4x)\,\delta(k-4) \end{bmatrix},$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 4x(2-x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} 0 \\ \frac{-4^3}{3!}x(2-x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_4 = \begin{bmatrix} \sin(4x) \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_5 = \begin{bmatrix} 0 \\ \frac{4^5}{5!}x(2-x) \end{bmatrix}, \dots$$

The inverse-VRDTs of the above vectors finally give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\left\{4y - \frac{(4y)^3}{3!} + \frac{(4y)^5}{5!} - \dots\right\} \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\sin(4y) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-F

In this problem, experimentation is made with initial data $\begin{bmatrix}0.001\\0.001\end{bmatrix}$ instead of homogeneous. We consider the inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix} + \begin{bmatrix}4y^3\sin(x)\\4y^3 + y^4\cos(x)\end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \tag{38}$$

subject to initial condition:

$$\begin{bmatrix}u(x,0)\\v(x,0)\end{bmatrix} = \begin{bmatrix}0.001\\0.001\end{bmatrix}, \quad x \in \mathbb{R}, \tag{39}$$

having exact solution vector given by:

$$\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix} = \begin{bmatrix}y^4\sin(x) + 0.001\\y^4 + 0.001\end{bmatrix}. \tag{40}$$

Now we use the VRDT method to solve this problem. The VRDT on (38) and (39) gives:

$$(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k + \begin{bmatrix}4\sin(x)\,\delta(k-3)\\4\delta(k-3) + \cos(x)\,\delta(k-4)\end{bmatrix}, \tag{41}$$

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0 = \begin{bmatrix}0.001\\0.001\end{bmatrix}. \tag{42}$$

Using iterative relations (41) and (42) we obtain the following vectors:

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0 = \begin{bmatrix}0.001\\0.001\end{bmatrix}, \begin{bmatrix}U(x)\\V(x)\end{bmatrix}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}U(x)\\V(x)\end{bmatrix}_2 = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}U(x)\\V(x)\end{bmatrix}_3 = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}U(x)\\V(x)\end{bmatrix}_4 = \begin{bmatrix}\sin(x)\\1\end{bmatrix}, \begin{bmatrix}U(x)\\V(x)\end{bmatrix}_5 = \begin{bmatrix}0\\0\end{bmatrix}, \ldots$$

The inverse-VRDTs of the above vectors finally give:

$$\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix} = \begin{bmatrix}\sum_{k=0}^{\infty} U_k(x)\, y^k\\\sum_{k=0}^{\infty} V_k(x)\, y^k\end{bmatrix} = \begin{bmatrix}y^4\sin(x) + 0.001\\y^4 + 0.001\end{bmatrix},$$

which is the exact solution vector.

It is noticed that all the model problems yield the VRDTM solution vectors in the form of infinite series whose compact forms are in agreement with the exact solutions. The results reveal that the technique is computationally less expensive in terms of mathematical manipulations as compared to other ones (for example, Adomian method, homotopy perturbation method and differential transform method). The VRDT method is a reliable and quite powerful technique that neither requires discretization nor perturbation.

Conclusion

For solving an initial-value problem for the Cauchy-Riemann system with analytic data, vectorial reduced differential transform (VRDT) method has been presented. The technique has been tested on a variety of homogeneous and inhomogeneous Cauchy-Riemann systems with various types of initial data. The VRDTM solution vector achieved, in each case, is in the form of infinite series whose compact form is in agreement with the exact solution vector. The technique is quite reliable and powerful.

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